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Solvability of a nonlocal boundary value problem for linear functional differential equations

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Abstract

In the paper, the problem on the existence and uniqueness of a solution to the nonlocal problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) = h(u) + c$$

is considered, where $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ and $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ are linear bounded operators, $q \in L([a,b];\mathbb{R})$, and $c \in \mathbb{R}$.

MSC: 34K06; 34K10

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1 Introduction and notation

On the interval $[a,b]$, we consider the boundary value problem

$$u'(t) = \ell(u)(t) + q(t), \tag{1}$$

$$u(a) = h(u) + c, \tag{2}$$

where $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ and $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ are linear bounded operators, $q \in L([a,b];\mathbb{R})$, and $c \in \mathbb{R}$. By a solution to the equation (1), we understand an absolutely continuous function $u : [a,b] \to \mathbb{R}$ satisfying equality (1) almost everywhere on the interval $[a,b]$. A solution to equation (1) satisfying the boundary condition (2) is said to be a solution to problem (1), (2).

The question on the solvability of various types of boundary value problems for functional differential equations and their systems is a classical topic in the theory of differential equations (see, *e.g.*, [1–16] and references therein). There is a lot of interesting general results, but only a few efficient conditions are known, namely, in the case where a nonlocal boundary condition is considered. In the present paper, new efficient conditions are found sufficient for the unique solvability of problem (1), (2). An important particular case of the boundary condition (2) is

$$u(a) = \lambda u(b) + c \tag{3}$$

with $\lambda \in \mathbb{R}$, which in turn contains the initial condition (if $\lambda = 0$), the periodic condition (if $\lambda = 1$), and the anti-periodic condition (if $\lambda = -1$). Problem (1), (3) is studied, *e.g.*, in [13, 17–19]. In [20, 21], the first step of our investigation in the general case was done. It is very useful to consider the boundary condition (2) as a nonlocal perturbation of the two-point condition (3). Therefore, we assume throughout the paper that the functional h is defined by the formula

$$h(v) \stackrel{\text{def}}{=} \lambda v(b) + h_0(v) - h_1(v) \quad \text{for } v \in C\big([a,b];\mathbb{R}\big), \tag{4}$$

where $\lambda > 0$ and $h_0, h_1 \in PF_{ab}$. There is no loss of generality in assuming this, because an arbitrary functional h can be represented in form (4).

The paper is organized as follows. Main results are formulated and proved in Section 2. In Section 3, the main results are applied to the equation with argument deviations

$$u'(t) = p(t)u\big(\tau(t)\big) - g(t)u\big(\mu(t)\big) + q(t), \tag{5}$$

where $p, g \in L([a,b];\mathbb{R}_+)$, $q \in L([a,b];\mathbb{R})$, and $\tau, \mu : [a,b] \to [a,b]$ are measurable functions. Some sufficient conditions for the validity of the inclusion $\ell \in \widetilde{V}_{ab}^{-}(h)$, which are part of the conditions for the main results, are given in Section 4.

The following notation is used throughout the paper:

1. $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.
2. $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $v : [a,b] \to \mathbb{R}$ endowed with the norm $\|v\|_C = \max\{|v(t)| : t \in [a,b]\}$.
3. $\widetilde{C}([a,b];D)$, where $D \subseteq \mathbb{R}$, is the set of absolutely continuous functions $v : [a,b] \to D$.
4. $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ endowed with the norm $\|p\|_L = \int_a^b |p(s)|\,ds$.
5. $L([a,b];D) = \{p \in L([a,b];\mathbb{R}) : p : [a,b] \to D\}$, where $D \subseteq \mathbb{R}$.
6. $C([a,b];D) = \{v \in C([a,b];\mathbb{R}) : v : [a,b] \to D\}$, where $D \subseteq \mathbb{R}$.
7. $\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$. P_{ab} is the set of operators $\ell \in \mathcal{L}_{ab}$, mapping the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.
8. F_{ab} is the set of linear bounded functionals $h : C([a,b];\mathbb{R}) \to \mathbb{R}$. PF_{ab} is the set of functionals $h \in F_{ab}$ mapping the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$.
9. $C_h([a,b];\mathbb{R}) = \{v \in C([a,b];\mathbb{R}) : v(a) = h(v)\}$, where $h \in F_{ab}$.

2 Main results

We assume throughout the paper that the following assumptions hold:

(H1) If $h(1) = 1$, then the operator ℓ is supposed to be 'nontrivial' in the sense that the condition $\ell(1) \not\equiv 0$ holds.

(H2) $\widetilde{h} \not\equiv 0$, where the functional $\widetilde{h}$ is defined by the formula $\widetilde{h}(v) = h(v) - v(a)$ for $v \in C([a,b];\mathbb{R})$.

Since we are interested in the unique solvability of problem (1), (2) for every q and c, both hypotheses (H1) and (H2) are rather natural. Indeed, if $\ell(1) \equiv 0$, then an arbitrary constant function is a solution to problem (1), (2) with $q \equiv 0$ and $c = 0$ in the case, where $h(1) = 1$. On the other hand, the assumption (H2) guarantees that the boundary condition (2) is not 'degenerated.'

Before formulation of the main results, we introduce the following definitions.

Definition 2.1 [22] Let $h \in F_{ab}$. An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set $\widetilde{V}_{ab}^{-}(h)$, if every function $u \in \widetilde{C}([a,b];\mathbb{R})$, satisfying the relations

$$u'(t) \geq \ell(u)(t) \quad \text{for a.e. } t \in [a,b], \qquad u(a) \geq h(u)$$

is nonpositive on the interval $[a,b]$.

Definition 2.2 [23] An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set $\mathcal{S}_{ab}(a)$ (resp. $\mathcal{S}_{ab}(b)$) if every function $u \in \widetilde{C}([a,b];\mathbb{R})$ satisfying the relations

$$u'(t) \geq \ell(u)(t) \quad \text{for a.e. } t \in [a,b], \qquad u(a) \geq 0 \qquad \bigl(\text{resp. } u(b) \leq 0\bigr)$$

is nonnegative (resp. nonpositive) on the interval $[a,b]$.

Remark 2.1 Efficient conditions, guaranteeing the validity of the inclusions $\ell \in \widetilde{V}_{ab}^{-}(h)$ and $\ell \in \mathcal{S}_{ab}(a)$, $\ell \in \mathcal{S}_{ab}(b)$, are stated, respectively, in [22] and [23].

2.1 Formulation of results

For the sake of transparency, we first formulate all the results; their proofs are postponed till Section 2.2 below.

Theorem 2.1 *Assume that there exist operators $\varphi_0 \in \widetilde{V}_{ab}^{-}(h)$ and $\varphi_1 \in P_{ab}$ such that the inequality*

$$\bigl|\ell(v)(t) - \varphi_0(v)(t)\bigr| \leq \varphi_1\bigl(|v|\bigr)(t) \quad \textit{for a.e. } t \in [a,b] \tag{6}$$

holds on the set $C_h([a,b];\mathbb{R})$. If, moreover,

$$\varphi_0 - \varphi_1 \in \widetilde{V}_{ab}^{-}(h), \tag{7}$$

then problem (1), (2) *has a unique solution.*

Corollary 2.1 *Let $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in P_{ab}$ and the relation $h(1) > 1$ hold. Moreover, there exists $\varepsilon \in [0, 1/2]$ such that*

$$\varepsilon\ell_0 \in \widetilde{V}_{ab}^{-}(h), \qquad -(1-2\varepsilon)\ell_0 - \ell_1 \in \widetilde{V}_{ab}^{-}(h). \tag{8}$$

Then problem (1), (2) *has a unique solution.*

Remark 2.2 Choosing a suitable number ε in Corollary 2.1 and using the results established in [22], we can obtain several efficient conditions, sufficient for the unique solvability of problem (1), (2). However, we do not formulate them in detail. We note only that for $\varepsilon = \frac{1}{2}$, the assumption (8) has the form

$$\frac{1}{2}\ell_0 \in \widetilde{V}_{ab}^{-}(h), \qquad -\ell_1 \in \widetilde{V}_{ab}^{-}(h).$$

Theorem 2.2 *Let there exist* $\varphi \in \widetilde{V}_{ab}^{-}(\omega)$ *such that the inequality*

$$\ell(v)(t)\operatorname{sgn} v(t) \geq \varphi(|v|)(t) \quad \text{for a.e. } t \in [a,b] \tag{9}$$

holds on the set $C_h([a,b];\mathbb{R})$, *where the functional* ω *is given by the formula*

$$\omega(v) \stackrel{\text{def}}{=} \lambda v(b) - h_0(v) - h_1(v) \quad \text{for } v \in C([a,b];\mathbb{R}). \tag{10}$$

Then problem (1), (2) *has a unique solution.*

Theorem 2.3 *Let* $\ell = \ell_0 - \ell_1$ *with* $\ell_0, \ell_1 \in P_{ab}$ *and the relations*

$$h(1) > 1, \qquad h_0(1) \leq 1 \tag{11}$$

be fulfilled. Moreover, there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *satisfying the conditions*

$$\gamma'(t) \leq -\ell_1(\gamma)(t) - \ell_0(1)(t) \quad \text{for a.e. } t \in [a,b], \tag{12}$$

$$\gamma(a) < h(\gamma), \tag{13}$$

$$\gamma(a) - \gamma(b) < \omega_1, \tag{14}$$

where

$$\omega_1 = 1 + \bigl(1 - h_0(1)\bigr)\min\left\{1, \frac{1}{\lambda}\right\} + 2\sqrt{\bigl(\lambda - h_1(1)\bigr)\min\left\{1, \frac{1}{\lambda}\right\}}. \tag{15}$$

Then problem (1), (2) *has a unique solution.*

Theorem 2.4 *Let* $\ell = \ell_0 - \ell_1$ *with* $\ell_0, \ell_1 \in P_{ab}$ *and the relations*

$$\lambda - h_1(1) > 1, \qquad h_0(1) > 0 \tag{16}$$

be fulfilled. Moreover, there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *such that condition* (12) *is satisfied and*

$$\gamma(a) \leq \lambda\gamma(b) - h_1(\gamma) - h_0(1), \tag{17}$$

$$\gamma(a) - \gamma(b) < \omega_2, \tag{18}$$

where

$$\omega_2 = 1 + \frac{1 - h_0(1)}{\lambda} + 2\sqrt{1 - \frac{1}{\lambda}h_1(1)}. \tag{19}$$

Then problem (1), (2) *has a unique solution.*

Remark 2.3 The assumption $h_0(1) \leq 1$ appearing in Theorem 2.3 is not supposed in Theorem 2.4. On the other hand, assumption (17) of Theorem 2.4 is stronger than assumption (13) of Theorem 2.3.

Theorem 2.5 *Let $\ell \in P_{ab}$, the relations*

$$h_0(1) \le 1, \qquad h_1(1) < \lambda \tag{20}$$

hold, and there exists a function $\gamma \in C([a,b];\mathbb{R})$ satisfying the conditions

$$\gamma'(t) \le \ell(\gamma)(t) \quad \text{for a.e. } t \in [a,b], \tag{21}$$

$$\gamma(a) < h(\gamma). \tag{22}$$

Let, moreover, at least one of the following conditions be fulfilled

(a)

$$\int_a^b \ell(1)(s)\,ds < \omega_1, \tag{23}$$

where the number ω_1 is given by formula (15);

(b)

$$\ell \in \mathcal{S}_{ab}(a); \tag{24}$$

(c)

$$\ell \in \mathcal{S}_{ab}(b). \tag{25}$$

Then problem (1), (2) *has a unique solution.*

Remark 2.4 If the relation $h(1) \ge 1$ is fulfilled, then the assumption concerning the existence of a function γ in Theorem 2.5 can be omitted. Indeed, since the operator ℓ is supposed to be nontrivial in the case where $h(1) = 1$, the function

$$\gamma(t) = 1 + \int_a^t \ell(1)(s)\,ds \quad \text{for } t \in [a,b]$$

satisfies conditions (21) and (22).

Remark 2.5 Define the operator $\varphi : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ by setting

$$\varphi(w)(t) \stackrel{\text{def}}{=} w(a+b-t) \quad \text{for } t \in [a,b], w \in C\big([a,b];\mathbb{R}\big).$$

Let

$$\hat{\ell}(w)(t) \stackrel{\text{def}}{=} -\ell\big(\varphi(w)\big)(a+b-t) \quad \text{for a.e. } t \in [a,b] \text{ and let all } w \in C\big([a,b];\mathbb{R}\big),$$

$$\hat{h}(w) \stackrel{\text{def}}{=} \frac{1}{\lambda}w(b) - \frac{1}{\lambda}h_0\big(\varphi(w)\big) + \frac{1}{\lambda}h_1\big(\varphi(w)\big) \quad \text{for } w \in C\big([a,b];\mathbb{R}\big),$$

$$\hat{q}(t) = -q(a+b-t) \quad \text{for a.e. } t \in [a,b], \qquad \hat{c} = -\frac{1}{\lambda}c.$$

It is not difficult to verify that if u is a solution to problem (1), (2), then the function $v \stackrel{\text{def}}{=} \varphi(u)$ is a solution to the problem

$$v'(t) = \hat{\ell}(v)(t) + \hat{q}(t), \qquad v(a) = \hat{h}(v) + \hat{c}, \tag{26}$$

and vice versa, if v is a solution to problem (26), then the function $u \stackrel{\text{def}}{=} \varphi(v)$ is a solution to problem (1), (2).

Using this transformation, we can immediately derive other conditions for the unique solvability of problem (1), (2), complementing those stated above. For example, Theorem 2.3 yields.

Theorem 2.3′ *Let $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in P_{ab}$ and the relations*

$$h(1) < 1, \qquad h_1(1) \le \lambda$$

be fulfilled. Let, moreover, there exist a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying the conditions

$$\gamma'(t) \ge \ell_0(\gamma)(t) + \ell_1(1)(t) \quad \textit{for a.e. } t \in [a,b],$$
$$\gamma(a) > h(\gamma),$$
$$\gamma(b) - \gamma(a) < \omega_3,$$

where

$$\omega_3 = 1 + \big(\lambda - h_1(1)\big)\min\left\{1, \frac{1}{\lambda}\right\} + 2\sqrt{\big(1 - h_0(1)\big)\min\left\{1, \frac{1}{\lambda}\right\}}.$$

Then problem (1), (2) *has a unique solution.*

2.2 Proofs

The following lemma is well known from the general theory of boundary value problems for functional differential equations (see, *e.g.*, [15, 24]; in the case, where the operator ℓ is strongly bounded, see also [1, 3, 14]).

Lemma 2.1 *Problem* (1), (2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$u'(t) = \ell(u)(t), \tag{27}$$
$$u(a) = h(u), \tag{28}$$

has only the trivial solution.

Remark 2.6 It follows immediately from Definition 2.1 and Lemma 2.1 that under the condition $\ell \in \widetilde{V}_{ab}^{-}(h)$ problem (1), (2) has a unique solution for every $q \in L([a,b];\mathbb{R})$ and $c \in \mathbb{R}$.

Now, we are in position to prove the main results. According to Lemma 2.1, it is sufficient to show that the homogeneous problem (27), (28) has only the trivial solution.

Proof of Theorem 2.1 Let u be a solution to problem (27), (28). Then, in view of (6), we get

$$\begin{aligned} u'(t) &= \varphi_0(u)(t) + \ell(u)(t) - \varphi_0(u)(t) \\ &\leq \varphi_0(u)(t) + \varphi_1\big(|u|\big)(t) \quad \text{for a.e. } t \in [a,b], \end{aligned} \tag{29}$$

$$\begin{aligned} u'(t) &= \varphi_0(u)(t) + \ell(u)(t) - \varphi_0(u)(t) \\ &\geq \varphi_0(u)(t) - \varphi_1\big(|u|\big)(t) \quad \text{for a.e. } t \in [a,b]. \end{aligned} \tag{30}$$

By virtue of the assumption $\varphi_0 \in \widetilde{V}_{ab}^{-}(h)$ and Remark 2.6, the problem

$$\alpha'(t) = \varphi_0(\alpha)(t) - \varphi_1\big(|u|\big)(t), \tag{31}$$

$$\alpha(a) = h(\alpha) \tag{32}$$

has a unique solution α. It follows from relations (29)-(31) that

$$\begin{aligned} (u+\alpha)'(t) &\leq \varphi_0(u+\alpha)(t) \quad \text{for a.e. } t \in [a,b], \\ (u-\alpha)'(t) &\geq \varphi_0(u-\alpha)(t) \quad \text{for a.e. } t \in [a,b]. \end{aligned} \tag{33}$$

On the other hand, conditions (28) and (32) yield

$$(u+\alpha)(a) = h(u+\alpha), \qquad (u-\alpha)(a) = h(u-\alpha). \tag{34}$$

Therefore, by virtue of the assumption $\varphi_0 \in \widetilde{V}_{ab}^{-}(h)$, relations (33) and (34) imply

$$\big|u(t)\big| \leq \alpha(t) \quad \text{for } t \in [a,b]. \tag{35}$$

Now, in view of (35) and the assumption $\varphi_1 \in P_{ab}$, we get from (31) the relation

$$\alpha'(t) \geq (\varphi_0 - \varphi_1)(\alpha)(t) \quad \text{for a.e. } t \in [a,b],$$

which, together with (7) and (32), yields that $\alpha(t) \leq 0$ for $t \in [a,b]$. Consequently, condition (35) guarantees $u \equiv 0$, and thus the homogeneous problem (27), (28) has only the trivial solution. □

Proof of Corollary 2.1 The validity of the corollary follows immediately from Theorem 2.1 with $\varphi_0 = \varepsilon\ell_0$ and $\varphi_1 = (1-\varepsilon)\ell_0 + \ell_1$. □

Proof of Theorem 2.2 Let u be a solution to problem (27), (28). Then, in view of (9), we get

$$\big|u(t)\big|' = \ell(u)(t)\operatorname{sgn} u(t) \geq \varphi\big(|u|\big)(t) \quad \text{for a.e. } t \in [a,b]. \tag{36}$$

On the other hand, by virtue of the assumptions $h_0, h_1 \in PF_{ab}$, condition (28) yields

$$\lambda\big|u(b)\big| = \big|u(a) - h_0(u) + h_1(u)\big| \leq \big|u(a)\big| + h_0\big(|u|\big) + h_1\big(|u|\big),$$

i.e.,

$$\big|u(a)\big| \geq \lambda\big|u(b)\big| - h_0\big(|u|\big) - h_1\big(|u|\big) = \omega\big(|u|\big). \tag{37}$$

Taking now the assumption $\varphi \in \widetilde{V}_{ab}^{-}(\omega)$ into account, we get from conditions (36) and (37) that

$$\big|u(t)\big| \leq 0 \quad \text{for } t \in [a,b].$$

Consequently, the homogeneous problem (27), (28) has only the trivial solution. □

Proof of Theorem 2.3 Suppose that problem (27), (28) possesses a nontrivial solution u. According to conditions (11)-(13) and the assumption $\ell_0 \in P_{ab}$, Proposition 4.2 guarantees the validity of the inclusion

$$-\ell_1 \in \widetilde{V}_{ab}^{-}(h).$$

Therefore, by virtue of the assumption $\ell_0 \in P_{ab}$, it follows from Definition 2.1 that u changes its sign. Put

$$M = \max\big\{u(t) : t \in [a,b]\big\}, \qquad m = -\min\big\{u(t) : t \in [a,b]\big\} \tag{38}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{39}$$

Obviously,

$$M > 0, \qquad m > 0, \tag{40}$$

and without loss of generality, we can assume that $t_m < t_M$. Using conditions (27), (28), (12), and (13), by virtue of (38), (40), and the assumption $\ell_0 \in P_{ab}$, we get

$$\begin{aligned}\big(M\gamma(t) + u(t)\big)' &\leq -\ell_1(M\gamma + u)(t) - \ell_0(M - u)(t)\\ &\leq -\ell_1(M\gamma + u)(t) \quad \text{for a.e. } t \in [a,b],\end{aligned} \tag{41}$$

$$M\gamma(a) + u(a) < h(M\gamma + u) \tag{42}$$

and

$$\begin{aligned}\big(m\gamma(t) - u(t)\big)' &\leq -\ell_1(m\gamma - u)(t) - \ell_0(m + u)(t)\\ &\leq -\ell_1(m\gamma - u)(t) \quad \text{for a.e. } t \in [a,b],\end{aligned} \tag{43}$$

$$m\gamma(a) - u(a) < h(m\gamma - u). \tag{44}$$

Hence, according to the condition $-\ell_1 \in \widetilde{V}_{ab}^-(h)$, inequalities (41)-(44) yield

$$M\gamma(t) + u(t) \geq 0, \qquad m\gamma(t) - u(t) \geq 0 \quad \text{for } t \in [a,b].$$

However, we assume that $\ell_1 \in P_{ab}$, and thus, it follows from (41) and (43) that

$$u'(t) \leq -M\gamma'(t), \qquad -u'(t) \leq -m\gamma'(t) \quad \text{for a.e. } t \in [a,b]. \tag{45}$$

The integration of the first inequality in (45) from t_m to t_M, in view of (39) and (40), implies

$$M + m \leq M\big(\gamma(t_m) - \gamma(t_M)\big),$$

i.e.,

$$0 < m \leq M\big(\gamma(t_m) - \gamma(t_M) - 1\big). \tag{46}$$

On the other hand, the integrations of the second inequality in (45) from a to t_m and from t_M to b, in view of (39) and (40), yield

$$u(a) + m \leq m\big(\gamma(a) - \gamma(t_m)\big), \qquad M - u(b) \leq m\big(\gamma(t_M) - \gamma(b)\big). \tag{47}$$

Moreover, on account of (38) and the assumptions $h_0, h_1 \in PF_{ab}$, condition (28) results in

$$u(a) - \lambda u(b) = h_0(u) - h_1(u) \geq -mh_0(1) - Mh_1(1).$$

Therefore, from (47) we get

$$M\big(\lambda - h_1(1)\big) + m\big(1 - h_0(1)\big) \leq m\big(\gamma(a) - \gamma(t_m) + \lambda\big(\gamma(t_M) - \gamma(b)\big)\big),$$

which, in view of (11) and (40), yields that

$$\begin{aligned} 0 &< M\big(\lambda - h_1(1)\big)\min\left\{1, \frac{1}{\lambda}\right\} \\ &\leq m\left(\gamma(a) - \gamma(t_m) + \gamma(t_M) - \gamma(b) - \big(1 - h_0(1)\big)\min\left\{1, \frac{1}{\lambda}\right\}\right). \end{aligned} \tag{48}$$

Now, from inequalities (46) and (48), we obtain

$$\gamma(a) - \gamma(b) > 1 + \big(1 - h_0(1)\big)\min\left\{1, \frac{1}{\lambda}\right\} \tag{49}$$

and

$$\begin{aligned} &\big(\lambda - h_1(1)\big)\min\left\{1, \frac{1}{\lambda}\right\} \\ &\quad \leq \big(\gamma(t_m) - \gamma(t_M) - 1\big) \\ &\qquad \times \left(\gamma(a) - \gamma(t_m) + \gamma(t_M) - \gamma(b) - \big(1 - h_0(1)\big)\min\left\{1, \frac{1}{\lambda}\right\}\right). \end{aligned} \tag{50}$$

In view of the inequality $4xy \le (x+y)^2$, it follows from condition (50) that

$$4\big(\lambda - h_1(1)\big)\min\left\{1, \frac{1}{\lambda}\right\} \le \left(\gamma(a) - \gamma(b) - 1 - \big(1 - h_0(1)\big)\min\left\{1, \frac{1}{\lambda}\right\}\right)^2,$$

which, together with (11) and (49), contradicts (14).

The contradiction obtained proves that problem (27), (28) has only the trivial solution. □

Proof of Theorem 2.4 Suppose that problem (27), (28) possesses a nontrivial solution u. According to conditions (12), (16), and (17) and the assumptions $\ell_0 \in P_{ab}$ and $h_0 \in PF_{ab}$, Proposition 4.2 guarantees the validity of the inclusion

$$-\ell_1 \in \widetilde{V}_{ab}^{-}\big(h^{-}\big),$$

where the functional h^- is defined by the formula

$$h^{-}(v) \stackrel{\text{def}}{=} \lambda v(b) - h_1(v) \quad \text{for } v \in C\big([a,b];\mathbb{R}\big). \tag{51}$$

Therefore, by virtue of the assumptions $\ell_0 \in P_{ab}$ and $h_0 \in PF_{ab}$, it follows from Definition 2.1 that u changes its sign. Define the numbers M and m by formulae (38), and choose $t_M, t_m \in [a,b]$ such that conditions (39) hold. Obviously, (40) is satisfied, and without loss of generality, we can assume that $t_m < t_M$. Using conditions (27), (28), (12), and (17), by virtue of (38), (40), (51), and the assumptions $\ell_0 \in P_{ab}$ and $h_0 \in PF_{ab}$, we get relations (41), (43),

$$M\gamma(a) + u(a) \le h^{-}(M\gamma + u) - h_0(M - u) \le h^{-}(M\gamma + u) \tag{52}$$

and

$$m\gamma(a) - u(a) \le h^{-}(m\gamma - u) - h_0(m + u) \le h^{-}(m\gamma - u). \tag{53}$$

Hence, according to the condition $-\ell_1 \in \widetilde{V}_{ab}^{-}(h^{-})$, inequalities (41), (43), (52), and (53) yield

$$M\gamma(t) + u(t) \ge 0, \qquad m\gamma(t) - u(t) \ge 0 \quad \text{for } t \in [a,b].$$

However, we assume that $\ell_1 \in P_{ab}$, and thus, it follows from (41) and (43) that inequalities (45) hold.

Now, analogously to the proof of Theorem 2.3, relations (49) and (50) can be derived. Since assumption (16) implies $\lambda > 1$, we get from (50) the inequality

$$4\left(1 - \frac{1}{\lambda}h_1(1)\right) \le \left(\gamma(a) - \gamma(b) - 1 - \frac{1 - h_0(1)}{\lambda}\right)^2,$$

which, together with (16) and (49), contradicts (18).

The contradiction obtained proves that problem (27), (28) has only the trivial solution. □

Proof of Theorem 2.5 Let u be a solution to problem (27), (28). We first show that each of assumptions (23), (24), or (25) ensures that u does not change its sign. Indeed, suppose that, on the contrary, u changes its sign. Define the numbers M and m by formulae (38), and choose $t_M, t_m \in [a,b]$ such that conditions (39) hold. Obviously, (40) is satisfied, and without loss of generality, we can assume that $t_M < t_m$.

(a) Let condition (23) hold. Then the integrations of (27) from a to t_M, from t_M to t_m, and from t_m to b, in view of (38), (39), and the assumption $\ell \in P_{ab}$, result in

$$M - u(a) = \int_a^{t_M} \ell(u)(s)\,ds \le M \int_a^{t_M} \ell(1)(s)\,ds, \tag{54}$$

$$M + m = -\int_{t_M}^{t_m} \ell(u)(s)\,ds \le m \int_{t_M}^{t_m} \ell(1)(s)\,ds, \tag{55}$$

$$u(b) + m = \int_{t_m}^{b} \ell(u)(s)\,ds \le M \int_{t_m}^{b} \ell(1)(s)\,ds. \tag{56}$$

Hence, by virtue of (40), condition (55) implies

$$0 < M \le m\left(\int_{t_M}^{t_m} \ell(1)(s)\,ds - 1\right). \tag{57}$$

On the other hand, on account of (38) and the assumptions $h_0, h_1 \in PF_{ab}$, condition (28) yields

$$\lambda u(b) - u(a) = h_1(u) - h_0(u) \ge -mh_1(1) - Mh_0(1).$$

Now, combining (54) and (56), we get

$$m\big(\lambda - h_1(1)\big) + M\big(1 - h_0(1)\big) \le M\left(\int_a^{t_M} \ell(1)(s)\,ds + \lambda \int_{t_m}^{b} \ell(1)(s)\,ds\right),$$

which, on account of (20) and (40), yields

$$\begin{aligned} 0 &< m\big(\lambda - h_1(1)\big) \min\left\{1, \frac{1}{\lambda}\right\} \\ &\le M\left(\int_I \ell(1)(s)\,ds - \big(1 - h_0(1)\big) \min\left\{1, \frac{1}{\lambda}\right\}\right), \end{aligned} \tag{58}$$

where $I = [a, t_M] \cup [t_m, b]$. Now, conditions (57) and (58) yield

$$\int_a^b \ell(1)(s)\,ds > 1 + \big(1 - h_0(1)\big) \min\left\{1, \frac{1}{\lambda}\right\} \tag{59}$$

and

$$\begin{aligned} \big(\lambda - h_1(1)\big) \min\left\{1, \frac{1}{\lambda}\right\} &\le \left(\int_{t_M}^{t_m} \ell(1)(s)\,ds - 1\right) \\ &\quad \times \left(\int_I \ell(1)(s)\,ds - \big(1 - h_0(1)\big) \min\left\{1, \frac{1}{\lambda}\right\}\right). \end{aligned} \tag{60}$$

In view of the inequality $4xy \leq (x+y)^2$, we get from condition (60) that

$$4\big(\lambda - h_1(1)\big)\min\left\{1, \frac{1}{\lambda}\right\} \leq \left(\int_a^b \ell(1)(s)\,ds - 1 - \big(1 - h_0(1)\big)\min\left\{1, \frac{1}{\lambda}\right\}\right)^2,$$

which, together with (20) and (59), contradicts (23).

(b) If (24) holds then, in view of Definition 2.2, the assumption $u(a) \geq 0$ (resp. $u(a) < 0$) implies $u(t) \geq 0$ (resp. $u(t) \leq 0$) for $t \in [a,b]$, which contradicts (40).

(c) If (25) holds, then, in view of Definition 2.2, the assumption $u(b) \geq 0$ (resp. $u(b) < 0$) implies $u(t) \geq 0$ (resp. $u(t) \leq 0$) for $t \in [a,b]$, which contradicts (40).

The contradictions obtained prove that u does not change its sign. We can assume without loss of generality, that the function u is nonnegative. Since $\ell \in P_{ab}$, it follows from equation (27) that

$$0 \leq u(a) \leq u(t) \leq u(b) \quad \text{for } t \in [a,b]. \tag{61}$$

Suppose that $u(b) > 0$. Then, in view of (20), (61), and the assumptions $h_0, h_1 \in PF_{ab}$, condition (28) yields

$$u(a) = \lambda u(b) + h_0(u) - h_1(u) \geq \big(\lambda - h_1(1)\big)u(b) > 0.$$

Hence, condition (61) implies

$$u(t) > 0 \quad \text{for } t \in [a,b]. \tag{62}$$

Put

$$v(t) = ru(t) - \gamma(t) \quad \text{for } t \in [a,b],$$

where

$$r = \max\left\{\frac{\gamma(t)}{u(t)} : t \in [a,b]\right\}.$$

According to (62), it is clear that

$$v(t) \geq 0 \quad \text{for } t \in [a,b] \tag{63}$$

and there exists $t_0 \in [a,b]$ such that

$$v(t_0) = 0. \tag{64}$$

Taking now (27), (21), (63), and the assumption $\ell \in P_{ab}$ into account, we obtain

$$v'(t) \geq \ell(v)(t) \geq 0 \quad \text{for a.e. } t \in [a,b].$$

Therefore, on account of conditions (63) and (64), the latter relation yields

$$0 = v(a) \leq v(t) \leq v(b) \quad \text{for } t \in [a,b]. \tag{65}$$

However, using (28), (20), (22), (65), and the assumptions $h_0, h_1 \in PF_{ab}$, we get the contradiction

$$0 = v(a) > \lambda v(b) + h_0(v) - h_1(v) \geq \bigl(\lambda - h_1(1)\bigr)v(b) \geq 0.$$

The contradiction obtained proves that $u(b) \leq 0$, and thus, condition (61) implies $u \equiv 0$. Consequently, the homogeneous problem (27), (28) has only the trivial solution. □

3 Differential equations with argument deviations

In this section, we give some corollaries of the main results for the equation with deviating arguments (5). Recall that we suppose that $p, g \in L([a,b];\mathbb{R}_+)$ and $\tau, \mu : [a,b] \to [a,b]$ are measurable functions. The conditions stated below show that problem (5), (2) is uniquely solvable, provided that either the coefficients p and g are 'small' in a certain sense, or the deviations τ and μ are 'close' to the identities (the functional differential equation (5) is 'close' to the ordinary one).

3.1 Formulation of results

Theorem 2.1 implies the following.

Corollary 3.1 *Let relations* (11) *be fulfilled, and let the functions p and τ satisfy at least one of the following conditions*:

(a)

$$\int_a^b p(s)\,ds \leq 2\bigl(1 - h_0(1)\bigr)\min\left\{1, \frac{1}{\lambda}\right\};$$

(b) $0 < h_0(1) < 1$, $\tau(t) \geq t$ *for a.e.* $t \in [a,b]$, *and*

$$\operatorname{ess\,sup}\left\{\int_t^{\tau(t)} p(s)\,ds : t \in [a,b]\right\} < \kappa^*, \tag{66}$$

where

$$\kappa^* = \sup\left\{\frac{\|p\|_L}{x}\ln\frac{2xe^x(1-h_0(1))}{\|p\|_L(e^x-1)} : 0 < x < \ln\frac{1}{h_0(1)}\right\}.$$

Let, moreover, the functions g and μ satisfy at least one of the following conditions:

(A)

$$\int_a^b g(s)\,ds < \frac{h(1)-1}{\lambda + h_0(1)}; \tag{67}$$

(B) $g \not\equiv 0$ *and*

$$\operatorname{ess\,sup}\left\{\int_{\mu(t)}^t g(s)\,ds : t \in [a,b]\right\} < \xi^*, \tag{68}$$

where

$$\xi^* = \sup\left\{\frac{\|g\|_L}{y}\ln\frac{ye^y(h(1)-1)}{\|g\|_L(e^y-1)(\lambda + h_0(1))} : 0 < y < \ln\frac{\lambda + h_0(1)}{1 + h_1(1)}\right\}. \tag{69}$$

Then problem (5), (2) *has a unique solution*.

From Theorem 2.3, we derive

Corollary 3.2 *Let relations* (11) *be fulfilled and*

$$\beta_0(a) < h(\beta_0), \tag{70}$$

$$\frac{h(1)-1}{\lambda - h_1(1)}\left(\frac{\beta_0(a)h_1(\beta_2)}{\lambda + h_0(1) - h_1(\beta_0)} + \beta_2(a)\right) < \omega_1(1-A_1), \tag{71}$$

where the number ω_1 *is given by formula* (15) *and*

$$A_1 = \frac{\beta_0(a)}{\lambda + h_0(1) - h_1(\beta_0)}\big(1 + h_1(\beta_1)\big) + \beta_1(a), \tag{72}$$

$$\beta_0(t) = \exp\left(\int_t^b g(s)\,ds\right) \quad \textit{for } t \in [a,b], \tag{73}$$

$$\beta_1(t) = \int_t^b g(s)\sigma(s)\left(\int_{\mu(s)}^s g(\xi)\,d\xi\right)\exp\left(\int_t^s g(\eta)\,d\eta\right)ds \quad \textit{for } t \in [a,b], \tag{74}$$

$$\beta_2(t) = \int_t^b g(s)\left(\int_{\mu(s)}^b p(\xi)\,d\xi\right)\exp\left(\int_t^s g(\eta)\,d\eta\right)ds + \int_t^b p(s)\,ds \quad \textit{for } t \in [a,b] \tag{75}$$

and

$$\sigma(t) = \frac{1}{2}\big(1 + \operatorname{sgn}\big(t - \mu(t)\big)\big) \quad \textit{for a.e. } t \in [a,b]. \tag{76}$$

Then problem (5), (2) *has a unique solution.*

Theorem 2.4 yields the following.

Corollary 3.3 *Let relations* (16) *be fulfilled,*

$$\beta_0(a) < \lambda - h_1(\beta_0) \tag{77}$$

and

$$\begin{aligned}&\frac{\lambda - h_1(1) - 1}{\lambda - h_1(1)}\left(\frac{\beta_0(a)}{\lambda - h_1(\beta_0)}\big(h_0(1) + h_1(\beta_2)\big) + \beta_2(a)\right)\\ &\quad < \left(\omega_2 + \frac{h_0(1)}{\lambda - h_1(1)}\right)(1 - A_2),\end{aligned} \tag{78}$$

where the functions β_0, β_1, β_2, *and* σ *are defined by formulae* (73)-(76), *the number* ω_2 *is given by formula* (19), *and*

$$A_2 = \frac{\beta_0(a)}{\lambda - h_1(\beta_0)}\big(1 + h_1(\beta_1)\big) + \beta_1(a). \tag{79}$$

Then problem (5), (2) *has a unique solution.*

Finally, we give statements concerning equation (5) with $g \equiv 0$, *i.e.*, the equation

$$u'(t) = p(t)u(\tau(t)) + q(t), \tag{80}$$

where $p \in L([a,b];\mathbb{R}_+)$, $q \in L([a,b];\mathbb{R})$, and $\tau : [a,b] \to [a,b]$ is a measurable function.
From Theorem 2.1 we can derive the following.

Corollary 3.4 *Let relations* (11) *be fulfilled,*

$$0 < \int_a^b p(s)\,ds \le 3\bigl(1 - h_0(1)\bigr)\min\left\{1, \frac{1}{\lambda}\right\} \tag{81}$$

and

$$\operatorname{ess\,sup}\left\{\int_{\tau(t)}^t p(s)\,ds : t \in [a,b]\right\} < \xi^*, \tag{82}$$

where

$$\xi^* = \sup\left\{\frac{\|p\|_L}{y}\ln\frac{3ye^y(h(1)-1)}{\|p\|_L(e^y-1)(\lambda + h_0(1))} : 0 < y < \ln\frac{\lambda + h_0(1)}{1 + h_1(1)}\right\}.$$

Then problem (80), (2) *has a unique solution.*

The next two statements follow from Theorem 2.5.

Corollary 3.5 *Let $p \not\equiv 0$, let the relations*

$$h(1) < 1, \qquad h_1(1) < \lambda$$

be fulfilled, and let

$$\operatorname{ess\,inf}\left\{\int_t^{\tau(t)} p(s)\,ds : t \in [a,b]\right\} > \xi_*, \tag{83}$$

where

$$\xi_* = \inf\left\{\frac{\|p\|_L}{y}\ln\frac{ye^y(1-h(1))}{\|p\|_L(e^y-1)(1-h_0(1))} : y > \ln\frac{1-h_0(1)}{\lambda - h_1(1)}\right\}. \tag{84}$$

Let, moreover,

$$\operatorname{ess\,sup}\left\{\int_t^{\tau(t)} p(s)\,ds : t \in [a,b]\right\} < \xi^*, \tag{85}$$

where

$$\xi^* = \sup\left\{\frac{\|p\|_L}{y}\ln\frac{ye^y}{\|p\|_L(e^y-1)} : y > 0\right\}. \tag{86}$$

Then problem (80), (2) *has a unique solution.*

Corollary 3.6 *Let* $p \not\equiv 0$, *let the relations*

$$h(1) \geq 1, \qquad h_0(1) \leq 1, \qquad h_1(1) < \lambda$$

be fulfilled, and let condition (85) *hold, where the number* ξ^* *is defined by formula* (86). *Then problem* (80), (2) *has a unique solution.*

3.2 Proofs

Proof of Corollary 3.1 Let the operators ℓ_0 and ℓ_1 be defined by the formulae

$$\ell_0(v)(t) \stackrel{\text{def}}{=} p(t)v\big(\tau(t)\big) \quad \text{for a.e. } t \in [a,b] \text{ and all } v \in C\big([a,b];\mathbb{R}\big) \tag{87}$$

and

$$\ell_1(v)(t) \stackrel{\text{def}}{=} g(t)v\big(\mu(t)\big) \quad \text{for a.e. } t \in [a,b] \text{ and all } v \in C\big([a,b];\mathbb{R}\big). \tag{88}$$

It is easy to verify that both conditions (a) and (b) of the corollary yield

$$\frac{1}{2}\ell_0 \in \widetilde{V}_{ab}^-(h)$$

(see Propositions 4.3 and 4.4).

On the other hand, both conditions (A) and (B) of the corollary guarantee the validity of the inclusion

$$-\ell_1 \in \widetilde{V}_{ab}^-(h)$$

(see Propositions 4.5 and 4.6).

Consequently, the assumptions of Corollary 2.1 are satisfied with $\varepsilon = \frac{1}{2}$. □

Proof of Corollary 3.2 Let the operators ℓ_0 and ℓ_1 be defined by formulae (87) and (88), respectively. According to condition (71), there exists $\varepsilon > 0$ such that

$$\frac{h(1)-1}{\lambda - h_1(1)}\left(\frac{\beta_0(a)(\varepsilon + h_1(\beta_2))}{\lambda + h_0(1) - h_1(\beta_0)} + \beta_2(a)\right) \leq \omega_1(1 - A_1). \tag{89}$$

Moreover, conditions (11), (70), and (71) imply $A_1 < 1$. Therefore, by virtue of (70) and (72), Proposition 4.7 guarantees the validity of the inclusion

$$-\ell_1 \in \widetilde{V}_{ab}^-(h).$$

Hence, according to Remark 2.6, the problem

$$\gamma'(t) = -g(t)\gamma\big(\mu(t)\big) - p(t), \tag{90}$$

$$\gamma(a) = h(\gamma) - \varepsilon \tag{91}$$

has a unique solution γ. It is clear that the function γ satisfies conditions (12) and (13). Using the inclusion $-\ell_1 \in \widetilde{V}_{ab}^{-}(h)$, we get $\gamma(t) \geq 0$ for $t \in [a,b]$, and thus, equation (90) yields

$$0 \leq \gamma(b) \leq \gamma(t) \leq \gamma(a) \quad \text{for } t \in [a,b]. \tag{92}$$

Furthermore, on account of (11), (92), and the assumptions $h_0, h_1 \in PF_{ab}$, condition (91) implies

$$\lambda\gamma(b) = \gamma(a) - h_0(\gamma) + h_1(\gamma) + \varepsilon > \gamma(a)\big(1 - h_0(1)\big) \geq 0.$$

Therefore, condition (92) yields that $\gamma(t) > 0$ for $t \in [a,b]$.

On the other hand, γ is a solution to the equation

$$\gamma'(t) = -g(t)\gamma(t) - g(t)\int_{\mu(t)}^{t} g(s)\gamma\big(\mu(s)\big)\,ds - g(t)\int_{\mu(t)}^{t} p(s)\,ds - p(t).$$

Hence, in view of notations (73) and (75), the Cauchy formula implies

$$\gamma(t) = \gamma(b)\beta_0(t) + \int_t^b g(s)\left(\int_{\mu(s)}^{s} g(\xi)\gamma\big(\mu(\xi)\big)\,d\xi\right)\exp\left(\int_t^s g(\eta)\,d\eta\right) + \beta_2(t)$$

for $t \in [a,b]$, whence we get

$$\gamma(t) \leq \gamma(b)\beta_0(t) + \gamma(a)\beta_1(t) + \beta_2(t) \quad \text{for } t \in [a,b]. \tag{93}$$

Taking now conditions (92), (93) and the assumptions $h_0, h_1 \in PF_{ab}$ into account, the relation (91) yields

$$\gamma(a) \geq \gamma(b)\big(\lambda + h_0(1) - h_1(\beta_0)\big) - \gamma(a)h_1(\beta_1) - h_1(\beta_2) - \varepsilon. \tag{94}$$

Therefore, we get from (93) and (94) the inequality

$$\gamma(a) \leq \gamma(a)A_1 + \frac{\beta_0(a)(\varepsilon + h_1(\beta_2))}{\lambda + h_0(1) - h_1(\beta_0)} + \beta_2(a). \tag{95}$$

On the other hand, by virtue of (92) and the assumptions $h_0, h_1 \in PF_{ab}$, condition (91) implies

$$\gamma(a) = \lambda\gamma(b) + h_0(\gamma) - h_1(\gamma) - \varepsilon < \big(\lambda - h_1(1)\big)\gamma(b) + \gamma(a)h_0(1),$$

and thus,

$$\gamma(a) - \gamma(b) < \frac{h(1) - 1}{\lambda - h_1(1)}\gamma(a). \tag{96}$$

Now, it is clear that conditions (89), (95), and (96) guarantee the validity of inequality (14).

Consequently, the assumptions of Theorem 2.3 are satisfied. □

Proof of Corollary 3.3 Let the operators ℓ_0 and ℓ_1 be defined by formulae (87) and (88), respectively. Condition (78) implies $A_2 < 1$. Therefore, according to (77) and (79), Proposition 4.7 guarantees the validity of the inclusion

$$-\ell_1 \in \widetilde{V}_{ab}^{-}\big(h^-\big), \tag{97}$$

where the functional h^- is defined by formula (51). Hence, by virtue of Remark 2.6, equation (90) has a unique solution γ satisfying the boundary condition

$$\gamma(a) = h^-(\gamma) - h_0(1). \tag{98}$$

It is clear that the function γ satisfies conditions (12) and (17). Using inclusion (97), we get $\gamma(t) \geq 0$ for $t \in [a,b]$, and thus, equation (90) yields the relation (92). Moreover, on account of (16), (92) and the assumption $h_1 \in PF_{ab}$, condition (98) implies

$$\lambda\gamma(b) = \gamma(a) + h_1(\gamma) + h_0(1) > 0.$$

Therefore, condition (92) yields that $\gamma(t) > 0$ for $t \in [a,b]$.

On the other hand, γ is a solution to the equation

$$\gamma'(t) = -g(t)\gamma(t) - g(t)\int_{\mu(t)}^{t} g(s)\gamma\big(\mu(s)\big)\,ds - g(t)\int_{\mu(t)}^{t} p(s)\,ds - p(t).$$

Hence, in view of notations (73) and (75), the Cauchy formula implies

$$\gamma(t) = \gamma(b)\beta_0(t) + \int_t^b g(s)\left(\int_{\mu(s)}^{s} g(\xi)\gamma\big(\mu(\xi)\big)\,d\xi\right)\exp\left(\int_t^s g(\eta)\,d\eta\right) + \beta_2(t)$$

for $t \in [a,b]$, whence we get relation (93). Taking now (93) and the assumption $h_1 \in PF_{ab}$ into account, condition (98) yields

$$\gamma(a) \geq \gamma(b)\big(\lambda - h_1(\beta_0)\big) - \gamma(a)h_1(\beta_1) - h_1(\beta_2) - h_0(1). \tag{99}$$

Therefore, we get from (93) and (99) the inequality

$$\gamma(a) \leq \gamma(a)A_2 + \frac{\beta_0(a)}{\lambda - h_1(\beta_0)}\big(h_0(1) + h_1(\beta_2)\big) + \beta_2(a). \tag{100}$$

On the other hand, by virtue of (92) and the assumption $h_1 \in PF_{ab}$, condition (98) implies

$$\gamma(a) = \lambda\gamma(b) - h_1(\gamma) - h_0(1) \leq \big(\lambda - h_1(1)\big)\gamma(b) - h_0(1),$$

and thus,

$$\gamma(a) - \gamma(b) \leq \frac{\lambda - h_1(1) - 1}{\lambda - h_1(1)}\gamma(a) - \frac{h_0(1)}{\lambda - h_1(1)}. \tag{101}$$

Now it is clear that conditions (78), (100), and (101) guarantee the validity of inequality (18).

Consequently, the assumptions of Theorem 2.4 are satisfied. □

Proof of Corollary 3.4 Let the operator ℓ_0 be defined by formula (87), and let $\ell_1 \equiv 0$. It is easy to verify that conditions (81) and (82) yield

$$\frac{1}{3}\ell_0 \in \widetilde{V}_{ab}^{-}(h), \qquad -\frac{1}{3}\ell_0 \in \widetilde{V}_{ab}^{-}(h)$$

(see Propositions 4.3 and 4.6).

Consequently, assumptions of Corollary 2.1 are satisfied with $\varepsilon = \frac{1}{3}$. □

Proof of Corollary 3.5 Let the operator ℓ be defined by the formula

$$\ell(v)(t) \stackrel{\text{def}}{=} p(t)v\big(\tau(t)\big) \quad \text{for a.e. } t \in [a,b] \text{ and all } v \in C\big([a,b];\mathbb{R}\big). \tag{102}$$

It is clear that $\ell \in P_{ab}$. Moreover, condition (85) implies the validity of inclusion (24) (see Proposition 4.8).

On the other hand, according to (83) and (84), there exist $y_0 > 0$ and $\varepsilon > 0$ such that

$$y_0 \geq \ln \frac{1 - h_0(1) + \varepsilon}{\lambda - h_1(1)} \tag{103}$$

and

$$\int_t^{\tau(t)} p(s)\,ds \geq \frac{\|p\|_L}{y_0} \ln \frac{y_0 e^{y_0}}{\|p\|_L(e^{y_0} + \delta)} \quad \text{for a.e. } t \in [a,b], \tag{104}$$

where

$$\delta = \frac{(\lambda - h_1(1))e^{y_0} + h_0(1) - 1 - \varepsilon}{1 - h(1)}. \tag{105}$$

Obviously, condition (103) yields $\delta \geq 0$. Therefore, we get from (104) the relation

$$e^{\frac{y_0}{\|p\|_L}\int_a^{\tau(t)} p(s)\,ds} + \delta \geq \frac{y_0}{\|p\|_L} e^{\frac{y_0}{\|p\|_L}\int_a^t p(s)\,ds} \quad \text{for a.e. } t \in [a,b]. \tag{106}$$

Now, we put

$$\gamma(t) = e^{\frac{y_0}{\|p\|_L}\int_a^t p(s)\,ds} + \delta \quad \text{for } t \in [a,b].$$

Then, by virtue of (105), (106), and the assumptions $h_0, h_1 \in PF_{ab}$, it is easy to verify that the function γ satisfies conditions (21) and (22).

Consequently, the assumptions of Theorem 2.5 are fulfilled. □

Proof of Corollary 3.6 Let the operator ℓ be defined by formula (102). It is clear that $\ell \in P_{ab}$. Moreover, condition (85) implies the validity of inclusion (24) (see Proposition 4.8).

Consequently, by virtue of Remark 2.4, the assumptions of Theorem 2.5 are satisfied. □

4 On the set $\widetilde{V}_{ab}^{-}(h)$

In this section, we give some sufficient conditions guaranteeing the inclusions $\ell \in \widetilde{V}_{ab}^{-}(h)$, $\ell \in \mathcal{S}_{ab}(a)$, and $\ell \in \mathcal{S}_{ab}(b)$, which are stated in [22, 23]. We first formulate rather general results.

Proposition 4.1 [22, Cor. 4.1] *Let $\ell \in P_{ab}$ be a b-Volterra operator, and let the functional h be defined by formula* (4), *where $\lambda > 0$ and $h_0, h_1 \in PF_{ab}$ are such that inequalities* (11) *are fulfilled. If there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying*

$$\gamma'(t) \geq \ell(\gamma)(t) \quad \text{for a.e. } t \in [a,b], \qquad \gamma(a) \geq h_0(\gamma),$$

then $\ell \in \widetilde{V}_{ab}^{-}(h)$.

Proposition 4.2 [22, Thms. 3.2 and 4.3] *Let $-\ell \in P_{ab}$, and let the functional h be defined by formula* (4), *where $\lambda > 0$ and $h_0, h_1 \in PF_{ab}$ are such that inequalities* (11) *are fulfilled. Then $\ell \in \widetilde{V}_{ab}^{-}(h)$ if and only if there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying*

$$\gamma'(t) \leq \ell(\gamma)(t) \quad \text{for a.e. } t \in [a,b], \qquad \gamma(a) < h(\gamma).$$

Choosing suitable functions γ in the propositions stated above, we can derive several efficient conditions sufficient for the validity of the inclusion $\ell \in \widetilde{V}_{ab}^{-}(h)$. These conditions are not formulated here in detail; we present, however, some of their corollaries for 'operators with argument deviations,' which are used in the proofs of the results stated in Section 3.

Proposition 4.3 [22, Cor. 5.3] *Let $p \in L([a,b];\mathbb{R}_+)$, $\tau : [a,b] \to [a,b]$ be a measurable function, and let the functional h be defined by formula* (4), *where $\lambda > 0$ and $h_0, h_1 \in PF_{ab}$ are such that inequalities* (11) *are fulfilled. If*

$$\int_a^b p(s)\,ds \leq \big(1 - h_0(1)\big) \min\left\{1, \frac{1}{\lambda}\right\},$$

then the operator ℓ, defined by formula (102), *belongs to the set $\widetilde{V}_{ab}^{-}(h)$.*

Proposition 4.4 [22, Thm. 5.3(c)] *Let $p \in L([a,b];\mathbb{R}_+)$, $\tau : [a,b] \to [a,b]$ be a measurable function, and let the functional h be defined by formula* (4), *where $\lambda > 0$ and $h_0, h_1 \in PF_{ab}$ are such that the inequalities*

$$h(1) > 1, \qquad 0 < h_0(1) < 1$$

are fulfilled. Assume that $\tau(t) \geq t$ for a.e. $t \in [a,b]$, and inequality (66) *holds, where*

$$\kappa^* = \sup\left\{ \frac{\|p\|_L}{x} \ln \frac{xe^x(1 - h_0(1))}{\|p\|_L(e^x - 1)} : 0 < x < \ln \frac{1}{h_0(1)} \right\}.$$

Then the operator ℓ, defined by formula (102), *belongs to the set $\widetilde{V}_{ab}^{-}(h)$.*

Proposition 4.5 [22, Rem. 4.3] *Let $g \in L([a,b];\mathbb{R}_+)$, $\mu : [a,b] \to [a,b]$ be a measurable function, and let the functional h be defined by formula* (4), *where $\lambda > 0$ and $h_0, h_1 \in PF_{ab}$ are such that inequalities* (11) *are fulfilled. If, moreover, inequality* (67) *is satisfied, then the operator ℓ, defined by the formula*

$$\ell(v)(t) \stackrel{\text{def}}{=} -g(t)v\bigl(\mu(t)\bigr) \quad \text{for a.e. } t \in [a,b] \text{ and all } v \in C\bigl([a,b];\mathbb{R}\bigr), \tag{107}$$

belongs to the set $\widetilde{V}_{ab}^{-}(h)$.

Proposition 4.6 [22, Cor. 5.2] *Let $g \in L([a,b];\mathbb{R}_+)$, $\mu : [a,b] \to [a,b]$ be a measurable function, and let the functional h be defined by formula* (4), *where $\lambda > 0$ and $h_0, h_1 \in PF_{ab}$ are such that inequalities* (11) *are fulfilled. If, moreover, $g \not\equiv 0$ and inequality* (68) *is satisfied, where the number ξ^* is given by formula* (69), *then the operator ℓ, defined by formula* (107), *belongs to the set $\widetilde{V}_{ab}^{-}(h)$.*

Proposition 4.7 [22, Thm. 5.7] *Let $g \in L([a,b];\mathbb{R}_+)$, $\mu : [a,b] \to [a,b]$ be a measurable function, and let the functional h be defined by formula* (4), *where $\lambda > 0$ and $h_0, h_1 \in PF_{ab}$ are such that inequalities* (11) *are fulfilled. If, moreover, inequalities* (70) *and*

$$\beta_0(a)\bigl(1 - h_1(\beta_1)\bigr) + h(\beta_0)\beta_1(a) < h(\beta_0)$$

are satisfied, where the functions β_0 and β_1 are defined by formulae (73), (74), *and* (76), *then the operator ℓ, defined by formula* (107), *belongs to the set $\widetilde{V}_{ab}^{-}(h)$.*

The last statement concerns the set $\mathcal{S}_{ab}(a)$.

Proposition 4.8 [23, Thm. 1.9] *Let $p \in L([a,b];\mathbb{R}_+)$, $p \not\equiv 0$, be such that inequality* (85) *is satisfied, where the number ξ^* is defined by formula* (86). *Then the operator ℓ, defined by formula* (102), *belongs to the set $\mathcal{S}_{ab}(a)$.*

Competing interests
The author declares that they have no competing interests.

Author's contributions
The author read and approved the final manuscript.

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